

DARBOUX TRANSFORMATION AND SOLUTIONS OF THE (2+1)-DIMENSIONAL SCHRÖDINGER-MAXWELL-BLOCH EQUATION

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Abstract

In this paper, we construct a Darboux transformation (DT) of the (2+1)-dimensional Schrödinger-Maxwell-Bloch equation (SMBE) which is integrable by the Inverse Scattering Method. Using this DT, the one-soliton solution and periodic solution are obtained from the "seed" solutions.

1 Introduction

It is well known that the nonlinear nature of the real system is considered to be fundamental in modern science. Nonlinearity is the fascinating subject which has many applications in almost all areas of science. Usually nonlinear phenomena are modeled by nonlinear ordinary and/or partial differential equations. Many of these nonlinear differential equations (NDE) are completely integrable. This means that these integrable NDE have some class interesting exact solutions such as solitons, dromions, rogue waves, similaritons and so on. They are of great mathematical as well as physical interest and the investigation of the solitons and other its sisters have become one of the most exciting and extremely active areas of research in modern science and technology in the past several decades. In particular, many of the completely integrable NDE are found and studied. Among of such integrable nonlinear systems the Schrödinger-Maxwell-Bloch equation (SMBE) plays an important role. The SMBE describe a soliton propagation in fibres with resonant and erbium-doped systems [1] and has the (1+1)-dimensions. In this paper our aim is to construct the Darboux transformation (DT) for the (2+1)-dimensional SMBE and finding its soliton solutions.

The paper is organized as follows. In Sec. 2, some main notations of our models are introduced. As an example, we consider the (1+1)-dimensional SMBE. The (2+1)-dimensional SMBE we present in Sec. 3. The DT of the (2+1)-dimensional SMBE we construct in Sec. 4. In the next Sec. 5, some exact solutions of the (2+1)-dimensional SMBE are given. Section 6 is devoted to conclusions.

2 The Schrödinger-Maxwell-Bloch equation in 1+1 dimensions

In this paper, we will study some properties of the (2+1)-dimensional SMBE. But in order to be self-contained, in this section we shall recall some important informations on the (1+1)-dimensional SMBE. In 1+1 dimensions, the SMBE is given by [2]

$$iq_t + q_{xx} + 2\delta|q|^2q - 2ip = 0, \quad (2.1)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (2.2)$$

$$\eta_x + \delta(q^*p + p^*q) = 0, \quad (2.3)$$

where q, p are complex functions, η is a real function, ω, δ are real constants ($\delta = \pm 1$) [3]. If $\delta = +1$ ($\delta = -1$) then we get the SMBE with the attractive (repulsive) interaction. This (1+1)-dimensional

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SMBE is integrable (see e.g. [5]-[6] and references therein). Its Lax pair has the form

$$\Psi_x = U\Psi, \quad (2.4)$$

$$\Psi_t = 2\lambda U\Psi + B\Psi, \quad (2.5)$$

where

$$U = -i\lambda\sigma_3 + U_0, \quad (2.6)$$

$$B = B_0 + \frac{i}{\lambda + \omega}B_{-1}. \quad (2.7)$$

Here

$$U_0 = \begin{pmatrix} 0 & q \\ -\delta q^* & 0 \end{pmatrix}, \quad (2.8)$$

$$B_0 = i\delta|q|^2\sigma_3 + i\begin{pmatrix} 0 & q_x \\ \delta q_x^* & 0 \end{pmatrix}, \quad (2.9)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -\delta p^* & -\eta \end{pmatrix}. \quad (2.10)$$

As well-known, the self-induced transparency effect in erbium doped nonlinear fibre in two-level resonant media describes by the SMBE [19,23]. In the literature, many types exact solutions of the SMB equation (2.1)-(2.3) have been found (see e.i. [4]).

Another interesting integrable system related with the SMBE (2.1)-(2.3) is the so-called M-XCIX equation. The M-XCIX equation reads as [3]

$$\mathbf{S}_t + \mathbf{S} \wedge \mathbf{S}_{xx} + \frac{2}{\omega} \mathbf{S} \wedge \mathbf{W} = 0, \quad (2.11)$$

$$\mathbf{W}_x + 2\omega \mathbf{S} \wedge \mathbf{W} = 0, \quad (2.12)$$

where \wedge denotes a vector product and

$$\mathbf{S} = (S_1, S_2, S_3), \quad \mathbf{W} = (W_1, W_2, W_3). \quad (2.13)$$

Here ω is a real function, $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1$, S_i and W_i are some real functions. The M-XCIX equation is integrable. Its Lax representation has the form

$$\Phi_x = U\Phi, \quad (2.14)$$

$$\Phi_t = V\Phi, \quad (2.15)$$

where

$$U = -i\lambda S, \quad (2.16)$$

$$V = \lambda^2 V_2 + \lambda V_1 + \frac{i}{\lambda + \omega} V_{-1} - \frac{i}{\omega} V_0. \quad (2.17)$$

Here

$$V_2 = -2iS, \quad (2.18)$$

$$V_1 = 0.5[S, S_x], \quad (2.19)$$

$$V_{-1} = V_0 = \begin{pmatrix} W_3 & W^+ \\ W^- & -W_3 \end{pmatrix}, \quad (2.20)$$

where

$$S = S_i \sigma_i = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad W = W_i \sigma_i = \begin{pmatrix} W_3 & W^+ \\ W^- & -W_3 \end{pmatrix}. \quad (2.21)$$

Here $S^\pm = S_1 \pm iS_2$, $W^\pm = W_1 \pm iW_2$, $[A, B] = AB - BA$, σ_i are Pauli matrices. With such U, V matrices, the equation

$$U_t - V_x + [U, V] = 0 \quad (2.22)$$

is equivalent to the equation

$$iS_t + 0.5[S, S_{xx}] + \frac{1}{\omega}[S, W] = 0, \quad (2.23)$$

$$iW_x + \omega[S, W] = 0, \quad (2.24)$$

It is nothing but the matrix form of the M-XCIX equation (2.11)-(2.12). Finally we recall that if $p = \eta = W_i = 0$ then the SMBE (2.1)-(2.3) and the M-XCIX equation (2.11)-(2.12) reduce to the NLSE

$$iq_t + q_{xx} + 2\delta|q|^2q = 0 \quad (2.25)$$

and to the Heisenberg ferromagnet equation

$$\mathbf{S}_t + \mathbf{S} \wedge \mathbf{S}_{xx} = 0, \quad (2.26)$$

respectively.

3 The (2+1)-dimensional Schrödinger-Maxwell-Bloch equation

Our aim in this paper is to construct the DT for the (2+1)-dimensional SMBE and using this to find some important its exact solutions. The (2+1)-dimensional SMBE reads as [28]

$$iq_t + q_{xy} - vq - 2ip = 0, \quad (3.1)$$

$$ir_t - r_{xy} + vr - 2ik = 0, \quad (3.2)$$

$$v_x + 2(rq)_y = 0, \quad (3.3)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (3.4)$$

$$k_x + 2i\omega k - 2\eta r = 0, \quad (3.5)$$

$$\eta_x + rp + kq = 0, \quad (3.6)$$

where q, k, r, v, p are complex functions, η is a real function, ω is the real constant. It is integrable by the IST. The corresponding Lax representation is given by

$$\Psi_x = A\Psi, \quad (3.7)$$

$$\Psi_t = 2\lambda\Psi_y + B\Psi, \quad (3.8)$$

where A and B have the form

$$A = -i\lambda\sigma_3 + A_0, \quad (3.9)$$

$$B = B_0 + \frac{i}{\lambda + \omega}B_{-1}. \quad (3.10)$$

Here

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (3.11)$$

$$B_0 = -0.5iv\sigma_3 + i \begin{pmatrix} 0 & q_y \\ r_y & 0 \end{pmatrix}, \quad (3.12)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \quad (3.13)$$

Let us now we consider the reduction $r = \delta q^*$, $k = \delta p^*$, where $*$ means a complex conjugate and δ is the real constant. Then the system (3.1)-(3.6) takes the form

$$iq_t + q_{xy} - vq - 2ip = 0, \quad (3.14)$$

$$v_x + 2\delta(|q|^2)_y = 0, \quad (3.15)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (3.16)$$

$$\eta_x + \delta(q^*p + p^*q) = 0. \quad (3.17)$$

Here ($\delta = \pm 1$) so that $\delta = +1$ corresponds to the attractive interaction and $\delta = -1$ to the repulsive interaction respectively. Let us we also present the spin system which is the equivalent counterpart of the (2+1)-dimensional SMBE (3.14)-(3.17). It looks like [28]

$$iS_t + \frac{1}{2}([S, S_y] + 2iuS)_x + \frac{1}{\omega}[S, W] = 0, \quad (3.18)$$

$$u_x I - \frac{i}{2}S[S_x, S_y] = 0, \quad (3.19)$$

$$iW_x + \omega[S, W] = 0 \quad (3.20)$$

or equivalently

$$iS_t + \frac{1}{2}[S, S_{xy}] + iuS_x + \frac{1}{\omega}[S, W] = 0, \quad (3.21)$$

$$u_x - \frac{i}{4}\text{tr}(S[S_x, S_y]) = 0, \quad (3.22)$$

$$iW_x + \omega[S, W] = 0. \quad (3.23)$$

It is the so-called ML-II equation [28]. The ML-II equation (3.18)-(3.20) is integrable by the IST. Its Lax representation can be written in the form

$$\Phi_x = U\Phi, \quad (3.24)$$

$$\Phi_t = 2\lambda\Phi_y + V\Phi. \quad (3.25)$$

Here the matrix operators U and V have the form

$$U = -i\lambda S, \quad (3.26)$$

$$V = \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W, \quad (3.27)$$

where

$$V_1 = 2Z = \frac{1}{2}([S, S_y] + 2iuS), \quad (3.28)$$

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \quad (3.29)$$

Some comments in order. First we recall that for these SMBE and ML-II equation, the spectral parameter obeys the equation: $\lambda_t = 2\lambda\lambda_y$ which has for example the following particular solution: $\lambda = (\beta_1 + \beta_2 y)(\beta_3 - 2\beta_1 t)^{-1}$, where β_j are in general some complex constants. Secondly, we also note that in 1+1 dimensions that is if $y = x$, both equations take the form of the (1+1)-dimensional SMBE (2.1)-(2.3) and the M-XCIX equation (2.11)-(2.12) respectively. Third, if $p = \eta = W_i = 0$ then the (2+1)-dimensional SMBE (2.1)-(2.3) and the ML-II equation (2.11)-(2.12) reduce to the (2+1)-dimensional nonlinear Schrödinger equation of the form

$$iq_t + q_{xy} - vq = 0, \quad (3.30)$$

$$v_x + 2\delta(|q|^2)_y = 0 \quad (3.31)$$

and to the M-I equation [31]

$$iS_t + \frac{1}{2}[S, S_{xy}] + iuS_x = 0, \quad (3.32)$$

$$u_x - \frac{i}{4}\text{tr}(S[S_x, S_y]) = 0, \quad (3.33)$$

respectively. Some properties of some integrable and notintegrable spin systems were studied in the refs. [11]-[36].

4 Darboux transformation

It is well-known that the DT has been proved to be an efficient way to find the exact solutions like solitons, dromions, positons, breathers, rogue wave solutions for integrable equations in 1+1 and 2+1 dimensions. In this section, considering the particularity of the Lax representation, we construct the DT of the (2+1)-dimensional SMBE (3.14)-(3.17). Furthermore, we will find some solutions of the (2+1)-dimensional SMBE using its DT.

4.1 One-fold DT

We consider the following transformation of Eq.(3.7)-(3.8)

$$\Psi' = T\Psi = (\lambda I - M)\Psi \quad (4.1)$$

such that

$$\Psi'_x = A'\Psi', \quad (4.2)$$

$$\Psi'_t = 2\lambda\Psi'_y + B'\Psi', \quad (4.3)$$

where A' and B' depend on q', v', p', η' and λ . Here

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.4)$$

The relation between $q', v', p', \eta', \lambda$ and $A' - B'$ is the same as the relation between q, v, p, η, λ and $A-B$. In order to hold Eqs.(4.2)-(4.3), the T must satisfies the following equations

$$T_x + TA = A'T, \quad (4.5)$$

$$T_t + TB = 2\lambda T_y + B'T. \quad (4.6)$$

Then the relation between q, v, p, η and q', v', p', η' can be reduced from these equations, which is in fact the DT of the (2+1)-dimensional SMBE. Comparing the coefficients of λ^i of the two sides of the equation (4.5), we get

$$\lambda^0 : M_x = A'_0 M - M A_0, \quad (4.7)$$

$$\lambda^1 : A'_0 = A_0 + i[M, \sigma_3], \quad (4.8)$$

$$\lambda^2 : iI\sigma_3 = i\sigma_3 I. \quad (4.9)$$

From (4.8) we obtain

$$q^{[1]} = q - 2im_{12}, \quad (4.10)$$

$$r^{[1]} = r - 2im_{21}. \quad (4.11)$$

or

$$q^{[1]} = q - 2im_{12}, \quad (4.12)$$

$$\delta q^{*[1]} = \delta q^* - 2im_{21}. \quad (4.13)$$

Hence we get $m_{21} = -m_{12}^*$ in our attractive interaction case that is if $\delta = +1$. Then comparing the coefficients of λ^i of the two sides of the equation (4.6) gives us

$$\lambda^0 : -M_t = iB'_0 - B'_0 M - iB_{-1} + MB_0, \quad (4.14)$$

$$\lambda^1 : 2M_y = B'_0 - B_0, \quad (4.15)$$

$$(\lambda + \omega)^{-1} : 0 = -i\omega B'_{-1} - iB'_{-1} M + i\omega B_{-1} + iMB_{-1}. \quad (4.16)$$

The last equation of this system gives

$$B'_{-1} = (M + \omega I)B_{-1}(M + \omega I)^{-1}. \quad (4.17)$$

At the same, from Eq.(4.15) we get

$$v' = v + 4im_{11y} = v - 4im_{22y}. \quad (4.18)$$

and hence we additionally have $m_{22} = m_{11}^*$. So the matrix M has the form

$$M = \begin{pmatrix} m_{11} & m_{12} \\ -m_{12}^* & m_{11}^* \end{pmatrix}, \quad M^{-1} = \frac{1}{|m_{11}|^2 + |m_{12}|^2} \begin{pmatrix} m_{11}^* & -m_{12} \\ m_{12}^* & m_{11} \end{pmatrix}, \quad (4.19)$$

$$M + \omega I = \begin{pmatrix} m_{11} + \omega & m_{12} \\ -m_{12}^* & \omega + m_{11}^* \end{pmatrix}, \quad (M + \omega I)^{-1} = \frac{1}{\square} \begin{pmatrix} m_{11}^* + \omega & -m_{12} \\ m_{12}^* & \omega + m_{11} \end{pmatrix}. \quad (4.20)$$

Here

$$\square = \det(M + \omega I) = \omega^2 + \omega(m_{11} + m_{11}^*) + |m_{11}|^2 + |m_{12}|^2. \quad (4.21)$$

The equation (4.17) gives

$$\eta' = \frac{(|\omega + m_{11}|^2 - |m_{12}|^2)\eta - pm_{12}^*(\omega + m_{11}) - p^*m_{12}(\omega + m_{11}^*)}{\square}, \quad (4.22)$$

$$p' = \frac{p(\omega + m_{11})^2 - p^*m_{12}^2 + 2\eta m_{12}(\omega + m_{11})}{\square}, \quad (4.23)$$

$$p^{*'} = \frac{p^*(\omega + m_{11}^*)^2 - pm_{12}^* + 2\eta m_{12}^*(\omega + m_{11}^*)}{\square}. \quad (4.24)$$

We now assume that

$$M = H\Lambda H^{-1}, \quad (4.25)$$

where

$$H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & \psi_1(\lambda_2; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_2(\lambda_2; t, x, y) \end{pmatrix}. \quad (4.26)$$

Here

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (4.27)$$

and $\det H \neq 0$, where λ_1 and λ_2 are complex constants. Using Eqs.(3.7)-(3.8), it is easy to get

$$H_x = -i\sigma_3 H \Lambda + A_0 H, \quad (4.28)$$

$$H_t = 2H_y \Lambda + B_0 H + B_{-1} H \Sigma, \quad (4.29)$$

where

$$\Sigma = \begin{pmatrix} \frac{i}{\lambda_1 + \omega} & 0 \\ 0 & \frac{i}{\lambda_2 + \omega} \end{pmatrix}. \quad (4.30)$$

In order to satisfy the constraints of S and B'_{-1} as mentioned above, we first notes that if $\delta = 1$ then

$$\Psi^+ = \Psi^{-1}, \quad A_0^+ = -A_0, \quad (4.31)$$

$$\lambda_2 = \lambda_1^*, \quad H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & -\psi_2^*(\lambda_1; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_1^*(\lambda_1; t, x, y) \end{pmatrix}, \quad (4.32)$$

$$H^{-1} = \frac{1}{\Delta} \begin{pmatrix} \psi_1^*(\lambda_1; t, x, y) & \psi_2^*(\lambda_1; t, x, y) \\ -\psi_2(\lambda_1; t, x, y) & \psi_1(\lambda_1; t, x, y) \end{pmatrix}. \quad (4.33)$$

So for the matrix M we have

$$M = \frac{1}{\Delta} \begin{pmatrix} \lambda_1|\psi_1|^2 + \lambda_2|\psi_2|^2 & (\lambda_1 - \lambda_2)\psi_1\psi_2^* \\ (\lambda_1 - \lambda_2)\psi_1^*\psi_2 & \lambda_1|\psi_2|^2 + \lambda_2|\psi_1|^2 \end{pmatrix}, \quad (4.34)$$

where

$$\Delta = |\psi_1|^2 + |\psi_2|^2. \quad (4.35)$$

Here we mention that $m_{22} = m_{11}^*$ and $m_{21} = -m_{12}^*$ that holds if $\lambda_2 = \lambda_1^*$. So finally we get the following DT of the (2+1)-dimensional SMBE:

$$q^{[1]} = q - 2im_{12} = q - \frac{2i(\lambda_1 - \lambda_2)\psi_1\psi_2^*}{\Delta}, \quad (4.36)$$

$$v^{[1]} = v + 4im_{11y} = v - 4i \left(\frac{\lambda_1|\psi_1|^2 + \lambda_2|\psi_2|^2}{\Delta} \right)_y, \quad (4.37)$$

$$\eta^{[1]} = \frac{(|\omega + m_{11}|^2 - |m_{12}|^2)\eta - pm_{12}^*(\omega + m_{11}) - p^*m_{12}(\omega + m_{11}^*)}{\square}, \quad (4.38)$$

$$p^{[1]} = \frac{p(\omega + m_{11})^2 - p^*m_{12}^2 + 2\eta m_{12}(\omega + m_{11})}{\square}, \quad (4.39)$$

$$p^{*[1]} = \frac{p^*(\omega + m_{11}^*)^2 - pm_{12}^{*2} + 2\eta m_{12}^*(\omega + m_{11}^*)}{\square}. \quad (4.40)$$

For our futher convenience, it is useful rewrite these formulas in the determinant form. First we have

$$m_{11} = \frac{\lambda_1|\psi_1|^2 + \lambda_2|\psi_2|^2}{\Delta} = \frac{\Delta_{11}}{\Delta}, \quad m_{12} = \frac{(\lambda_1 - \lambda_2)\psi_1\psi_2^*}{\Delta} = \frac{\Delta_{12}}{\Delta}, \quad (4.41)$$

where

$$\Delta_{11} = \det \begin{pmatrix} \psi_1 & -\lambda_2\psi_2^* \\ \psi_2 & \lambda_1\psi_1^* \end{pmatrix}, \quad \Delta_{12} = -\det \begin{pmatrix} \psi_1 & \lambda_1\psi_1 \\ \psi_2^* & \lambda_2\psi_2^* \end{pmatrix}. \quad (4.42)$$

Finally we note that we can rewrite the above presented solution in terms of the elements of the matrix T . From

$$T = \lambda I - M = \lambda I + t_0^{[1]} = \frac{1}{\Delta_1} \begin{pmatrix} T_{111} & T_{112} \\ T_{121} & T_{122} \end{pmatrix} \quad (4.43)$$

follows that

$$M = \begin{pmatrix} m_{11} & m_{12} \\ -m_{12}^* & m_{11}^* \end{pmatrix} = \lambda I - T = -t_0^{[1]}. \quad (4.44)$$

Finally we can write the 1-sd solution of the (2+1)-dimensional SMBE which follows from the corresponding one-fold DT, as

$$q^{[1]} = q + 2i(t_0^1)_{12}, \quad (4.45)$$

$$v^{[1]} = v - 4i(t_0^1)_{11y}, \quad (4.46)$$

$$\eta^{[1]} = \frac{(|\omega - (t_0^1)_{11}|^2 - |(t_0^1)_{12}|^2)\eta + p(t_0^1)_{12}^*(\omega - (t_0^1)_{11}) + p^*(t_0^1)_{12}(\omega - (t_0^1)_{11}^*)}{\square}, \quad (4.47)$$

$$p^{[1]} = \frac{p(\omega - (t_0^1)_{11})^2 + p^*(t_0^1)_{12}^2 - 2\eta(t_0^1)_{12}(\omega - (t_0^1)_{11})}{\square}, \quad (4.48)$$

$$p^{*[1]} = \frac{p^*(\omega - (t_0^1)_{11}^*)^2 + p(t_0^1)_{12}^{*2} - 2\eta(t_0^1)_{12}^*(\omega - (t_0^1)_{11}^*)}{\square}. \quad (4.49)$$

4.2 N-fold DT

To construct the n -fold DT of the (2+1)-dimensional SMBE, we introduce n eigenfunctions [5]

$$\begin{pmatrix} \Phi_{1,i} \\ \Phi_{2,i} \end{pmatrix} = \Phi|_{\lambda=\lambda_i}, \quad i = 1, 2, \dots, 2n$$

where λ_i and Φ_i satisfy the conditions: $\lambda_{2n-1} = \lambda_{2n}^*$ and $\Phi_{2,2n} = \Phi_{1,2n-1}^*$, $\Phi_{2,2n-1} = -\Phi_{1,2n}^*$. The n -fold DT of the (2+1)-dimensional SMBE can be written as (these formulas are same as for the Hirota-Maxwell-Bloch equation) [5]:

$$T_{nt} + T_n U = U^{[n]} T_n, \quad (4.50)$$

$$T_{nz} + T_n V = V^{[n]} T_n. \quad (4.51)$$

Hence we obtain

$$U_0^{[n]} = U_0 + i[\sigma_3, t_{n-1}^{[n]}], \quad (4.52)$$

$$V_{-1}^{[n]} = T_n|_{\lambda=-\omega} V_{-1} T_n^{-1}|_{\lambda=-\omega}. \quad (4.53)$$

The matrix T_n can be expand as [5]:

$$T_n(\lambda; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \lambda_{2n}) = \lambda^n I + t_{n-1}^{[n]} \lambda^{n-1} + \dots + t_1^{[n]} \lambda + t_0^{[n]} = \frac{1}{\Delta_n} \begin{pmatrix} T_{n11} & T_{n12} \\ T_{n21} & T_{n22} \end{pmatrix}, \quad (4.54)$$

where

$$\Delta_n = \begin{vmatrix} \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} \end{vmatrix} \quad (4.55)$$

$$T_{n11} = \begin{vmatrix} 1 & 0 & \lambda & 0 & \dots & \lambda^{n-1} & 0 & \lambda^n \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} & \lambda_1^n \Phi_{1,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} & \lambda_2^n \Phi_{1,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} & \lambda_3^n \Phi_{1,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} & \lambda_4^n \Phi_{1,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} & \lambda_{2n-1}^n \Phi_{1,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} & \lambda_{2n}^n \Phi_{1,2n} \end{vmatrix} \quad (4.56)$$

$$T_{n12} = \begin{vmatrix} 0 & 1 & 0 & \lambda & \dots & 0 & \lambda^{n-1} & 0 \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} & \lambda_1^n \Phi_{1,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} & \lambda_2^n \Phi_{1,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} & \lambda_3^n \Phi_{1,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} & \lambda_4^n \Phi_{1,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} & \lambda_{2n-1}^n \Phi_{1,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} & \lambda_{2n}^n \Phi_{1,2n} \end{vmatrix} \quad (4.57)$$

$$T_{n21} = \begin{vmatrix} 1 & 0 & \lambda & 0 & \dots & \lambda^{n-1} & 0 & 0 \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} & \lambda_1^n \Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} & \lambda_2^n \Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} & \lambda_3^n \Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} & \lambda_4^n \Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} & \lambda_{2n-1}^n \Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} & \lambda_{2n}^n \Phi_{2,2n} \end{vmatrix} \quad (4.58)$$

$$T_{n22} = \begin{vmatrix} 0 & 1 & 0 & \lambda & \dots & 0 & \lambda^{n-1} & \lambda^n \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} & \lambda_1^n \Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} & \lambda_2^n \Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} & \lambda_3^n \Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} & \lambda_4^n \Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} & \lambda_{2n-1}^n \Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} & \lambda_{2n}^n \Phi_{2,2n} \end{vmatrix}. \quad (4.59)$$

Using the n -fold DT, finally we can give the determinant form of the n -th solution of the (2+1)-dimensional SMBE. It is given by

$$q^{[n]} = q + 2i(t_{n-1}^{[n]})_{12}, \quad (4.60)$$

$$v^{[1]} = v - 4i(t_{n-1}^{[n]})_{11y}, \quad (4.61)$$

$$p^{[n]} = \frac{2\eta T_{n11}T_{n12} - p^*T_{n12}^2 + pT_{n11}^2}{T_{n11}T_{n22} - T_{n12}T_{n21}}|_{\lambda=-\omega}, \quad (4.62)$$

$$\eta^{[n]} = \frac{\eta(T_{n11}T_{n22} + T_{n12}T_{n21}) - p^*T_{n12}T_{n22} + pT_{n11}T_{n21}}{T_{n11}T_{n22} - T_{n12}T_{n21}}|_{\lambda=-\omega}. \quad (4.63)$$

Here we considered the n -fold DT of the (2+1)-dimensional SMBE. In the next sections, we apply this DT to construct some exact solutions of this equation.

5 Solutions

5.1 Periodic solutions

First let us consider a periodic solution of the (2+1)-dimensional SMBE. To do it, as a seed solution, we take the following its solution $q = de^{i\rho}$, $v = m$, $p = ifq$, $\eta = 1$. Here $\rho = ax + by + ct$ and a, b, c, m, f, d are some constants. Then the corresponding eigenfunctions are given by (see i.e. Refs.[4]-[10])

$$\psi_1(\lambda; x, y, t) = de^{\frac{i}{2}\rho + ic(\lambda)}, \quad (5.1)$$

$$\psi_2(\lambda; x, y, t) = (i(c_1\lambda + \frac{b}{2}) - \lambda)e^{(-\frac{i}{2}\rho + ic(\lambda))}, \quad (5.2)$$

where $c(\lambda) = c_1(\lambda)t + c_2(\lambda)x + c_3(\lambda)y$. Now using Eqs.(5.1)-(5.2) we can write the periodic solution of the (2+1)-dimensional SMBE corresponding to the one-fold DT.

5.2 Soliton solutions

To get the one-soliton solution we take the seed solution as $q = 0$, $v = 0$, $p = 0$, $\eta = 1$. Let $\lambda_1 = a + bi$. Then the corresponding associated linear system takes the form

$$\Psi_{1x} = -i\lambda\Psi_1, \quad (5.3)$$

$$\Psi_{2x} = i\lambda\Psi_2, \quad (5.4)$$

$$\Psi_{1t} = 2\lambda\Psi_{1y} + \frac{i}{\lambda + \omega}\Psi_1, \quad (5.5)$$

$$\Psi_{2t} = 2\lambda\Psi_{2y} - \frac{i}{\lambda + \omega}\Psi_2. \quad (5.6)$$

This system admits the following exact solutions

$$\Psi_1 = \Psi_{10}e^{-i\lambda_1 x + i\mu_1 y + i(2\lambda_1\mu_1 + \frac{1}{\lambda_1 + \omega})t}, \quad (5.7)$$

$$\Psi_2 = \Psi_{20}e^{i\lambda_1 x - i\mu_1 y - i(2\lambda_1\mu_1 + \frac{1}{\lambda_1 + \omega})t}, \quad (5.8)$$

or

$$\Psi_1 = e^{-i\lambda_1 x + i\mu_1 y + i(2\lambda_1\mu_1 + \frac{1}{\lambda_1 + \omega})t + \delta_1 + i\delta_2}, \quad (5.9)$$

$$\Psi_2 = e^{i\lambda_1 x - i\mu_1 y - i(2\lambda_1\mu_1 + \frac{1}{\lambda_1 + \omega})t - \delta_1 - i\delta_2 + i\delta_0}, \quad (5.10)$$

where $\mu_1 = c + id$, δ_i and c, d are real constants. Then the one-soliton solution of the (2+1)-dimensional SMBE is given by

$$q^{[1]} = \frac{-4aie^{L1}}{e^{N1} + e^{N2}}, \quad (5.11)$$

$$v^{[1]} = v - 4i(t_0^1)_{11y}, \quad (5.12)$$

$$p^{[1]} = \frac{4ae^{L1}[(\lambda_1 + \omega)e^{N1} - (\lambda_1^* - \omega)e^{N2}]}{(e^{N1} + e^{N2})^2}, \quad (5.13)$$

$$\eta^{[1]} = \frac{4a^2e^{L1+L2}[(\lambda_1 + \omega)e^{N1} - (\lambda_1^* - \omega)e^{N2}]}{(e^{N1} + e^{N2})^2} \frac{[(\lambda_1 + \omega)e^{N2} - (\lambda_1^* - \omega)e^{N1}]}{(e^{N1} + e^{N2})^2}, \quad (5.14)$$

where

$$L1 = 2bx - 2dy - 4(bc + ad)t + \frac{2(a+\omega)}{(a-bi+\omega)(a+bi+\omega)}ti + \delta_0i;$$

$$L2 = -2bx + 2dy - 4(bc + ad)t - \frac{2(a+\omega)}{(a-bi+\omega)(a+bi+\omega)}ti + \delta_0i;$$

$$N1 = -2ax + 2cyi + 4(ac - bd)ti + \frac{2(a+\omega)}{(a-bi+\omega)(a+bi+\omega)}ti + 2(\delta_1 + \delta_2i);$$

$$N2 = 2ax - 2cyi - (4(ac - bd)ti + \frac{2(a+\omega)}{(a-bi+\omega)(a+bi+\omega)}ti - 2(\delta_1 + \delta_2i - \delta_0i)).$$

Using the above presented n -fold DT, similarly we can construct the n -soliton solution of the (2+1)-dimensional SMBE.

6 Conclusion

In this paper, we have constructed the DT for the (2+1)-dimensional SMBE. Using the derived DT, some exact solutions like, one-soliton solution and periodic solution are obtained. The determinant representations of the obtained solutions of the (2+1)-dimensional SMBE are given. Using the above presented results, one can also construct the n -solitons, breathers and rogue wave solutions of the (2+1)-dimensional SMBE. It is interesting to note that rogue wave solutions of nonlinear equations is currently one of the hottest topics in nonlinear physics and mathematics. The application of the obtained solutions in physics will be one interesting subject. In particular, we hope that the presented solutions may find some usages in experiment or optical fibre communication. Also we note that we will study some important generalizations of the (2+1)-dimensional SMBE in future.

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